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On the application of Karhunen–Loève transform to transient dynamic systems

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ABSTRACT

The Karhunen–Loève transform (KLT) has become a popular method in various fields of engineering science. Due to its ability to identify the most prominent features in the underlying system dynamics the KLT is a favorable method for such tasks as process monitoring, model order reduction or optimum control. However, it is a well-known fact that the KLT is 'case sensitive'. That is that changes in the dynamic system behavior can decisively affect the KLT results. As much as this property is desired for monitoring problems, it limits the performance of KLT in model order reduction or optimum control problems, if systems are subject to structural changes.

Recent research interest focuses on extending applications of KLT to systems with transient dynamic behavior or changing boundary conditions. Approaches have been published that circumvent the limitations of KLT by either assuming reasonable comparability of system dynamics or by measuring the representative performance of KLT-bases a posteriori. However, such methods require additional simulations of the full size system and thus jeopardize the idea of model order reduction.

In this paper, we introduce a novel a priori measure to evaluate the performance of the current KLT-basis. This procedure can be of great help in either monitoring or adaptive control of systems that show intermittent transient and (quasi-)stationary dynamic behavior. This a priori measure prepares the path for adaptive model order reduction schemes. Moreover, it can be used to measure the stationarity of multi-dimensional dynamic processes.

1. Introduction

The Karhunen–Loève transform (KLT)[1] has become a well-established method in various fields of scientific research ranging from biological, meteorological and seismological to engineering applications (we refer to [1,2] for review articles). In mechanical engineering the KLT has become a very popular especially for process monitoring [2–6], model order reduction [7–11] and control problems [12,13]. In any application the KLT is used to capture the most prominent dynamic behavior neglecting degrees of freedom of vanishing contribution. This is done by truncating the KL-series expansion of the

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[1] Also known as proper orthogonal decomposition (POD), principal component analysis (PCA) or singular value decomposition (SVD).

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time-dependent, spatially distributed vector function $y : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbf{y}(t) \in \mathscr{Y} \subset \mathbb{R}^n$ as in

$$\mathbf{y}(t) = \bar{\mathbf{y}} + \sum_{i=1}^{n} \alpha_i(t)\boldsymbol{\psi}_i, \quad \alpha_i : \mathbb{N} \rightarrow \alpha_i(k) \in \mathbb{R}, \ \boldsymbol{\psi} \in \mathbb{R}^n, \tag{1}$$

at $i = m < n$. $\bar{\mathbf{y}}$ the is vector of mean-values, $\boldsymbol{\psi}_i$ are the spatially distributed 'characteristic functions',[2] α_i are the corresponding time-dependent 'weight factors' and $n \in \mathbb{N}$ is the number of degrees of freedom. As a result the system dynamics are projected onto a submanifold spanned by the reduced basis $\tilde{\Psi} = \{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_m\} \subset \mathscr{Y}$ of characteristic functions.

It can be shown that the reduced KLT-basis $\tilde{\Psi}$ is the optimum reduced basis to describe the dynamic behavior of $\mathbf{y}(t)$ in least square sense (Ref. e.g. to [12]). It is also well known that the KLT is case-sensitive, since the characteristic functions not only represent the structural system properties, but also those of the excitation. This effect has been studied by Ma et al. [14] who compare the characteristic functions derived from numerical simulation results to those first $\boldsymbol{\psi}_i$ calculated from experimental measurement data: Although the underlying systems seem to be comparable, due to model uncertainties and slight differences in the excitations, only the very first characteristic functions show reasonable agreement between simulation and measurement. Starting with the second characteristic functions the results differ significantly.

Recently, this restriction has challenged scientists to push forward applications of KLT into the field of transient dynamics and changing boundary conditions/excitation features, respectively. In an attempt to capture both transient and stationary parts in wave motion (standing and traveling waves) Feeny [15] discusses the complex orthogonal decomposition (COD) as a generalization of KLT. In his work he extends real oscillators to complex 'phasors' by Fourier or Hilbert transform to preserve the phase information during the spatial decomposition. Ma et al. [16] introduce an a posteriori criterion for the approximation performance of a KLT-reduced order simulation of transient coupled oscillator dynamics. In a recent publication Buffoni et al. [17] encounter problems with a nonlinear observer for unsteady three-dimensional flows based on KLT. The first characteristic functions fail to give an accurate representation of the flow field. The authors suggest to increase the amount of data for KLT to improve the accuracy of the reduced basis $\tilde{\Psi}$.

The majority of current approaches to extend the reduced KLT-basis representation to transient system dynamics share a lack of information and raise the principal question: Equipped with a reduced KLT-basis $\tilde{\Psi}^Q$ derived from previous data Y^Q, how do we know whether $\tilde{\Psi}^Q$ is still optimal or needs to be updated. In this case, it would be favorable to have a method to relate the approximation performance of a given KLT-basis to the current system dynamics $\mathbf{y}(y)$. In an approach, Homescu et al. [18] combine small sample statistical condition estimation (SCE) method, adjoint method for error estimation and a perturbation-ansatz to estimate regions of validity for POD-reduced models and the approximation error. In spite of being an a priori method, this approach not only requires to solve the adjoint problem of the initial differential equation but also bears uncertainty of the derived error-estimates that are subject to probability distributions and may deviate from the effective approximation errors by magnitudes.

In an attempt to solve these problems, we introduce an a priori measure to associate the current underlying system dynamics with the given KLT-basis. This procedure can be of great help in either monitoring or adaptive control of systems that show intermittent transient and (quasi-)stationary dynamic behavior. It could prepare the path for efficient adaptive model reduction schemes. Moreover, it could be used to measure the stationarity of multidimensional dynamic processes.

This paper is outlined as follows: In Section 2 we briefly introduce the basic properties of KLT. In Section 4 we discuss the issue of stationarity with the focus on KLT. In Section 5 we introduce a new approach of a priori measurement. In Section 6 we display examples for the effect of the new measure. Finally we summarize our findings in Section 7.

2. Basics of KLT

In this section, we sketch the mathematical background of KLT with emphasis on the part that introduces the case-sensitivity to the transform. Throughout the paper we consider discretized time $t = kT$ and for the sake of simplicity assume zero-mean $\bar{\mathbf{y}} = \mathbf{0}$ unless stated otherwise.

3. Calculation of the KLT-basis

Following Karhunen [19], we are looking for a transform as in Eq. (1) and require uncorrelated weight factors:

$$E\{\alpha_i(k)\alpha_j(k)\} = \lambda_j \delta_{ij}, \quad i, j = 1, \ldots, n, \tag{2}$$

where $\lambda_j \in \mathbb{R} \geq 0$ are unknown scalars and Kronecker's function δ_{ij}. The decorrelation of weight factors α_i corresponds to orthogonal characteristic functions. Thus, supposing normalized characteristic functions, $\boldsymbol{\Psi}^{-1} = \boldsymbol{\Psi}^{\mathrm{T}}$ and forming the vector of weight factors $\boldsymbol{\alpha}(k) := [\alpha_1(k) \ \alpha_2(k) \ \cdots \ \alpha_n(k)]^{\mathrm{T}}$ and transfer matrix $\boldsymbol{\Psi} := [\boldsymbol{\psi}_1 \ \boldsymbol{\psi}_2 \ \cdots \ \boldsymbol{\psi}_n]$ we can rewrite Eq. (1) as

$$\boldsymbol{\alpha}(k) = \boldsymbol{\Psi}^{-1}\mathbf{y}(k). \tag{3}$$

[2] Also referred to as the 'KL-modes', 'nonlinear eigenmodes' or 'empirical eigenmodes'.

Combining Eqs. (2), (3) and Eq. (3) leads to

$$E\{\boldsymbol{\psi}_i^{\mathrm{T}}\mathbf{y}\mathbf{y}^{\mathrm{T}}\boldsymbol{\psi}_j\} = \lambda_j\delta_{ij}, \quad i,j = 1,\ldots,n. \tag{4}$$

Extracting time independent $\boldsymbol{\psi}_{i,j}$ from estimate $E\{\cdot,\cdot\}$ and regarding $\mathbf{y}(k)$ as vector random process gives

$$\boldsymbol{\psi}_i^{\mathrm{T}}\mathbf{C}_{yy}\boldsymbol{\psi}_j = \lambda_j\delta_{ij}, \quad i,j = 1,2,\ldots,n. \tag{5}$$

that is satisfied by solutions $\boldsymbol{\psi}_i$, $i = 1,2,\ldots,n$, of eigenvalue problem

$$\mathbf{C}_{yy}\boldsymbol{\psi}_i = \lambda_i\boldsymbol{\psi}_i, \quad i = 1,2,\ldots,n, \tag{6}$$

since $\boldsymbol{\psi}_i^{\mathrm{T}}\boldsymbol{\psi}_j = \delta_{ij}$, with covariance matrix $\mathbf{C}_{yy} = E\{\mathbf{y}\mathbf{y}^{\mathrm{T}}\}$ of $\mathbf{y}(k)$.

Eq. (6) gives n eigenvalues λ_i and n eigenvectors $\boldsymbol{\psi}_i$ that can be normalized by the Gram–Schmidt procedure. While basis Ψ spans vector space $\mathscr{Y} \subset \mathbb{R}^n$ of system dynamics $\mathbf{y}$ completely, eigenvalues λ_i represent the average energy contribution of the corresponding characteristic function $\boldsymbol{\psi}_i$ to $\mathbf{y}(k)$. The original KLT is unique and in case of discrete signals a lossless coordinate transform. For reduced order representation of $\mathbf{y}(k)$, characteristic functions $\boldsymbol{\psi}_i$ are ordered according to the corresponding eigenvalues λ_i. Then, series expansion of $\mathbf{y}(k)$ in (1) is truncated according to a desired portion of system energy. In general, the motion represented by the very first few weight factors $\alpha_i,\ i = 1,\ 2,\ \ldots,\ \ell \ll n$, cover more than 95 percent of the signal power/kinetic energy of the system.

3.1. Case-sensitivity of the KLT

The KLT is 'signal dependent', because the characteristic functions are derived from Covariance matrix $\mathbf{C}_{yy}$ by (6) that contains the second order statistical properties of $\mathbf{y}(k)$. In practice, $\mathbf{C}_{yy}$ is calculated from data Y^{Q} gathered over a fixed time interval $k \in I^{\mathrm{Q}} = [a^{\mathrm{Q}}\ \ b^{\mathrm{Q}}]$. Thus, structural changes in the dynamical pattern of $\mathbf{y}(k)$ affect $\mathbf{C}_{yy}$ leading to different sets of characteristic functions $\boldsymbol{\psi}_i^P$ if (6) was updated at a later interval $I^P = [a^P\ \ b^P]$.

Nevertheless, if the underlying dynamics of $\mathbf{y}(k)$ change Eq. (1) still holds true. However, the principal motion of the system may not occur along the previously chosen reduced set of characteristic functions $\tilde{\Psi}^{\mathrm{Q}} = \{\boldsymbol{\psi}_1^{\mathrm{Q}}, \boldsymbol{\psi}_2^{\mathrm{Q}}, \ldots, \boldsymbol{\psi}_\ell^{\mathrm{Q}}\}$ any more. Thus, the initial reduced basis $\tilde{\Psi}^{\mathrm{Q}}$ may not be the optimal basis any longer. In order to represent the same portion of kinetic energy of the current dynamics of $\mathbf{y}(k)$ basis $\tilde{\Psi}^{\mathrm{Q}}$ has to be extended to consider a larger number $m, \ell < m \leqslant n$ of characteristic functions. Of course this step would reduce the computational advantage of the KLT reduced order model. Therefore, the better choice might be to update basis Ψ^{Q}.

4. On the performance of the KLT-basis

4.1. Stationarity in the sense of KLT

In order to determine the performance of a given reduced KLT-basis $\tilde{\Psi}^{\mathrm{Q}}$ to represent $\mathbf{y}(k)$ in the presence of changing boundary conditions a priori, we have to relate the current dynamics to the set of reference data $Y^{\mathrm{Q}} = \{\mathbf{y}(k) | k \in I^{\mathrm{Q}} = [a_q, b_q]\}$[3] used to derive Ψ^{Q}. In order to judge whether basis $\tilde{\Psi}^{\mathrm{Q}}$ still is optimal or has to be updated, we need to detect changes in the dynamic behavior of $\mathbf{y}(k)$ that affect $\mathbf{C}_{yy}$. Thus, we have to compare the statistical properties of the system dynamics during the reference interval I^{Q} with the present statistical properties of $\mathbf{y}(k)$.[4]

Due to

$$\begin{aligned}\Delta c_{ij} &= c_{ij}^{N+1} - c_{ij}^{N}\\ &= \frac{1}{N+1}\sum_{k=1}^{N+1} y_i(k)y_j(k) - \frac{1}{N}\sum_{k=1}^{N} y_i(k)y_j(k)\\ &= \frac{1}{N(N+1)}\sum_{k=1}^{N} y_i(k)y_j(k) + \frac{1}{N+1}y_i(N+1)y_j(N+1)\end{aligned}$$

leading to

$$\lim_{N\to\infty}\Delta c_{ij} = 0, \tag{7}$$

differences of the kind of $\Delta\mathbf{C}_{yy} = \mathbf{C}_{yy}^{N+1} - \mathbf{C}_{yy}^{N}$, where $\mathbf{C}_{yy}^{N+1}$ and $\mathbf{C}_{yy}^{N}$ are based on intervals $I^{N+1} = [1, N+1]$ and $I^N = [1, N]$, are of little help for monitoring the statistical properties.

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[3] We apply Sirovich's 'method of snapshots' [20] to accumulate data sets Y.

[4] The use of subintervals is quite common in time series analysis, Refs. e.g. [21,22].

As a consequence, we compare the covariance matrix of $\mathbf{y}(k)$ of reference interval I^Q

$$\mathbf{C}_{yy}^Q = \frac{1}{b_q - (a_q + 1)} \sum_{k=a_q}^{b_q} \mathbf{y}(k)\mathbf{y}^{\mathrm{T}}(k), \tag{8}$$

with the covariance matrix

$$\mathbf{C}_{yy}^P = \frac{1}{b_p - (a_p + 1)} \sum_{k=a_p}^{b_p} \mathbf{y}(k)\,\mathbf{y}^{\mathrm{T}}(k) \tag{9}$$

calculated from most recent data $Y^P = \{\mathbf{y}(k) \mid k \in I^P = [a_p, b_p]\}$. It is tempting to demand the covariance of current dynamics $\mathbf{C}_{yy}^P$ to comply with

$$\mathbf{C}_{yy}^P \approx \mathbf{C}_{yy}^Q \tag{10}$$

to assure that $\tilde{\Psi}^Q$ remains the optimal basis for the current values of $\mathbf{y}(k)$.

Note that requiring constant covariance matrix and considering the assumption of constant mean $\bar{\mathbf{y}}$ corresponds to the definition of 'weak stationarity' as in e.g. [23]:

$$\bar{y}_i(q) = \frac{1}{b_q - (a_q - 1)} \sum_{k=a_q}^{b_q} y_i(k) = \text{const.},$$

$$c_{ij}(q) = \frac{1}{b_q - (a_q - 1)} \sum_{k=a_q}^{b_q} (y_i(k) - \bar{y}_i)(y_j(k) - \bar{y}_j) = \text{const.}, \tag{11}$$

for all time shifts $q \in \mathbb{N}$.

However, since KLT-basis Ψ consists of normalized characteristic functions, a given reduced basis $\tilde{\Psi}^Q$ remains the optimal representation of $\mathbf{y}(k)$ if

$$\mathbf{C}_{yy}^P \approx \vartheta \cdot \mathbf{C}_{yy}^Q \tag{12}$$

holds, where $\vartheta \in \mathbb{R}$ is arbitrary, $\mathbf{C}_{yy}^P$ is based on recent data gathered from time interval I^P. Thus, we can loosen requirement (10) and take (12) to define the 'stationarity in the sense of KLT'.

5. Measure of stationarity

For application, we need to find a practical way to evaluate Eq. (12). Instead of comparing covariance matrices C_{yy}^Q, C_{yy}^P element wise, we find it favorable to have a single coefficient. Therefore, we take advantage of the symmetry of $\mathbf{C}_{yy}$ and define representative vector

$$\mathbf{v} := [c_{11} \;\; c_{12} \;\; \ldots \;\; c_{1n} \;\; c_{22} \;\; c_{23} \;\; \ldots \;\; c_{2n} \;\; \ldots \;\; c_{nn}]^{\mathrm{T}} \tag{13}$$

that resembles the upper triangular matrix of C_{yy} including the diagonal elements. As a metric, we choose the correlation function and define the performance coefficient

$$\theta := \frac{\mathbf{v}^{Q^{\mathrm{T}}} \mathbf{v}^P}{\|\mathbf{v}^Q\| \|\mathbf{v}^P\|}. \tag{14}$$

Thus, we can compare the statistical properties of $\mathbf{y}(k)$ of two intervals I^Q, I^P of arbitrary sizes N^Q, N^P and arbitrary time shift $\Delta k = a_P - a_Q$.

Interpretation: Comparing the statistical properties of stationary system behavior with that of transient behavior θ becomes a measure of stationarity. Thus, we can monitor the performance of $\tilde{\Psi}^Q$ based on data Y^Q to represent the current dynamics of $\mathbf{y}(k)$, $k \in I^P$ without updating KLT. Measure θ has a range of $[-1, 1]$. Values of θ close to 1 resemble good similarity, values close to 0 resemble poor similarity in the statistical properties.

5.1. Application

In a KLT-based model order reduction scheme, we define a tolerance region $[\theta_{\lim}, 1]$ with lower bound $\theta_{\lim}$. Thus, we can use a given reduced basis $\tilde{\Psi}_Q$ as long as $\theta \in [\theta_{\lim}, 1]$. We can control the approximation accuracy of the reduced order model by choice of limit $\theta_{\lim}$.

Remark. Focusing on transient dynamical systems we face a dilemma: On the one hand, there is a minimum length for time intervals I^Q, I^P to capture representative sets of data Y^Q and Y^P. On the other hand, there is a maximum length for intervals I^Q, I^P to precisely detect structural changes (transients) in the dynamical behavior of $\mathbf{y}(k)$. The minimum interval length $T_{\min}$ can be calculated for stationary systems, only, by measuring the convergence of series $\mathbf{C}_{yy}(b)$ (e.g. in Eq. (8)) as upper bound b increases. For transient systems, we have to predefine the minimum interval length $T_{\min}$ and assume that data of the corresponding interval will be representative. Thus, we can only detect one or the other: We can either calculate minimum interval length, assuming stationarity of $\mathbf{y}(k)$ or we can detect changes in the dynamics of $\mathbf{y}(k)$, assuming to have chosen the right interval length.

In our investigation we focused on linear and nonlinear oscillators. Therefore, we took Shannon's theorem as a reference and defined $T_{\min} = 2\max\{\tau_{\max i}\}$, with maximum period time $\tau_{\max}$ of coordinate $y_i(k)$.

5.1.1. Detecting the onset of stationarity

In cases of transient dynamics followed by stationary motion, it is of interest to detect the onset of stationarity. Examples are monitoring problems, where different dynamic states shall be detected and characterized, e.g. [24] or adaptive model reduction applications, where the number of full size model simulations necessary for calculating KLT-basis Ψ shall be minimized.

Since we do not have any reference or starting point, we need to calculate covariance matrices $\mathbf{C}_{yy}^{Q}$ and $\mathbf{C}_{yy}^{P}$ based on two floating intervals of minimum size $T_{\min}$:

$$\mathbf{C}_{yy}^{Q}(k) = \frac{1}{T_{\min}} \sum_{\kappa=k-\Delta k-(1/2)T_{\min}}^{k-\Delta k+(1/2)T_{\min}} \mathbf{y}(\kappa)\mathbf{y}^{\mathrm{T}}(\kappa) \tag{15}$$

and

$$\mathbf{C}_{yy}^{P}(k) = \frac{1}{T_{\min}} \sum_{\kappa=k-T_{\min}}^{k} \mathbf{y}(\kappa)\mathbf{y}^{\mathrm{T}}(\kappa), \tag{16}$$

with recent simulation/observation time k. According to the definition of weak stationarity, the choice of time lag Δk is arbitrary. Thus we set $\Delta k = T_{\min}$. The onset of stationary dynamic behavior is characterized by the requirement:

$$a_{\mathrm{stat}} = \min\{k \mid \theta(k) \in [\theta_{\mathrm{lim}1}, 1]\}. \tag{17}$$

5.1.2. Detecting the decay of stationarity

Of course, it is equally important to detect the decay of stationary dynamic system behavior. For this, we simply change the intervals of the covariance matrices: Matrix $\mathbf{C}_{yy}^{Q}$ stays constant and is evaluated over fixed interval $I^{Q} = [a_{\mathrm{stat}} - \Delta k - \frac{1}{2}T_{\min}, a_{\mathrm{stat}} - \Delta k + \frac{1}{2}T_{\min}]$ at the beginning of stationary behavior a_{stat}, while variable matrix $\mathbf{C}_{yy}^{P}$ is based on moving interval $I^{P} = [k - T_{\min}, k]$. We calculate

$$\mathbf{C}_{yy}^{Q}(k) = \frac{1}{T_{\min}} \sum_{\kappa=a_{\mathrm{stat}}-\Delta k-(1/2)T_{\min}}^{a_{\mathrm{stat}}-\Delta k+(1/2)T_{\min}} \mathbf{y}(\kappa)\mathbf{y}^{\mathrm{T}}(\kappa) \tag{18}$$

and

$$\mathbf{C}_{yy}^{P}(k) = \frac{1}{T_{\min}} \sum_{\kappa=k-T_{\min}}^{k} \mathbf{y}(\kappa)\mathbf{y}^{\mathrm{T}}(\kappa). \tag{19}$$

The interval of stationary system behavior ends at

$$b_{\mathrm{stat}} = \max\{k \mid \theta(k) \in [\theta_{\mathrm{lim}2}, 1]\}. \tag{20}$$

Note that, in order to avoid switching effects of θ, due to the change of calculation method for $\mathbf{C}_{yy}^{Q}$ from (15) to (18) we use different tolerances $\theta_{\mathrm{lim}2} < \theta_{\mathrm{lim}1}$ for determining the onset and the decay of stationarity. Thus, for chosen tolerances $\theta_{\mathrm{lim}1}$ and $\theta_{\mathrm{lim}2}$ dates a_{stat} and b_{stat} describe the limits of time interval $I^{\mathrm{stat}} = [a_{\mathrm{stat}}, b_{\mathrm{stat}}]$ where covariance matrices $\mathbf{C}_{yy}^{Q}$ and $\mathbf{C}_{yy}^{\mathrm{stat}}$ are sufficiently similar. Thus, reduced basis $\tilde{\Psi}^{Q}$ representing Y^{Q} can be used to simulate $\mathbf{y}(k)$ while $k \in I^{\mathrm{stat}}$. We will show the benefit of this method in the following section.

6. Exemplary studies

Coupled oscillators can exhibit various different types of dynamic behavior: traveling waves, intermediate transient dynamics and stationary motion in the form of standing waves. They are suitable systems to study the performance of the stationarity measure θ, Eq. (14), to detect the onset and decay of stationary dynamics. In parallel we apply the stationarity measure θ to analyze a triple pendulum as an example of highly nonlinear systems.

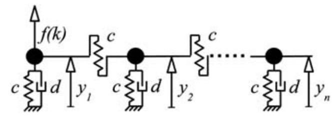

Fig. 1. Schematics of coupled oscillator.

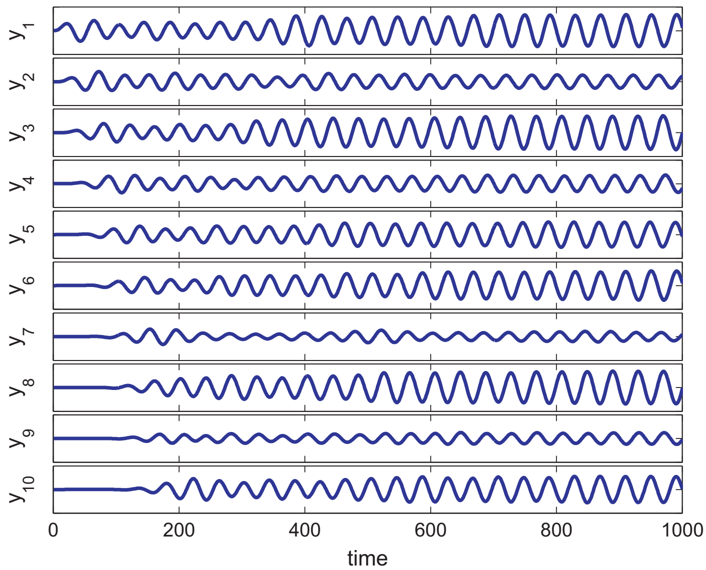

Fig. 2. Simulated dynamics of coupled oscillator: begin of oscillation.

6.1. *Transient dynamics of the coupled oscillator*

We study the effect of changing system dynamics on KLT-basis Ψ using simulation of the coupled oscillator as shown in Fig. 1, exemplarily. The oscillator consists of 10 point masses $m = 1$ with coordinates: $y_1, y_2, \ldots, y_{10}$, suspended by linear springs $c = 1$, subject to viscose damping $d = 1$. The masses are connected by springs $c = 1$. The system is driven by a sinusoidal force $f(k) = f_0(k) = 2\sin(0.9\sqrt{3c}\ k)$ acting at mass y_1.

6.1.1. *Initiation of stationary oscillations*

With initial conditions $\{\mathbf{y}_0 = \mathbf{0}, \dot{\mathbf{y}}_0 = \mathbf{0}\}$ the coupled oscillator undergoes three different stages of dynamic behavior: At the beginning, a group of waves travels from y_1 to y_{10} for $k = [0, 190]$, followed by intermediate transient behavior for $k = [190, 310]$. Finally, after $k = 310$, the system settles for stationary dynamics and forms a standing wave, see Fig. 2.

We analyze the dynamics of the coupled oscillator by calculating KLT-bases Ψ for consecutive intervals I, $T_{\min} = 80$. Due to the case-sensitivity of KLT, we find different sets of characteristic functions $\boldsymbol{\psi}_i$ and corresponding eigenvalues λ_i, $i = 1, 2, 10$ for each interval I during the period of transient system dynamics. We require a level of similarity of 98 percent in two consecutive KLT-bases Ψ^Q and Ψ^P to call process $\mathbf{y}(k)$ stationary across $I^Q \cap I^P$. Thus, we define limits $\theta_{\lim 1} = 0.98$ and $\theta_{\lim 2} = 0.98^2 \approx 0.96$.

We measure the similarity of bases Ψ^Q and Ψ^P of any two consecutive intervals I^Q and I^P by (1) correlation coefficient $r\{\boldsymbol{\psi}_i^Q, \boldsymbol{\psi}_i^P\}$[5] and by (2) measure θ, applying Eqs. (15), (16), (17) and Eqs. (18), (19) and (20), respectively. The correlation coefficients $r\{\boldsymbol{\psi}_i^Q, \boldsymbol{\psi}_i^P\}$ of the first three characteristic functions are shown in Fig. 4 for the beginning oscillations. Fig. 5 shows the relative energy $(\lambda_i / \sum \lambda_i)$ represented by the first three characteristic functions $\boldsymbol{\psi}_i^P$ vs. time and the stationary measure θ is displayed in Fig. 3. Vertical dashed lines describe the detected beginning whereas, vertical solid lines thedetected end of stationarity.

Comparing θ with $r\{\boldsymbol{\psi}_i^Q, \boldsymbol{\psi}_i^P\}$ we can find a reasonable correlation between the stationarity measure θ and the correlation coefficients $r\{\boldsymbol{\psi}_i^Q, \boldsymbol{\psi}_i^P\}$ of those characteristic functions representing significant portions of the system's energy: In Figs. 3 and 4 we can observe that both the stationarity measure θ as well as the correlation coefficients $r\{\boldsymbol{\psi}_1^Q, \boldsymbol{\psi}_1^P\}$ and $r\{\boldsymbol{\psi}_2^Q, \boldsymbol{\psi}_2^P\}$

[5] For this analysis, the order of consecutive characteristic functions ψ_i^Q and ψ_i^P was rearranged for the best match of corresponding characteristic functions.

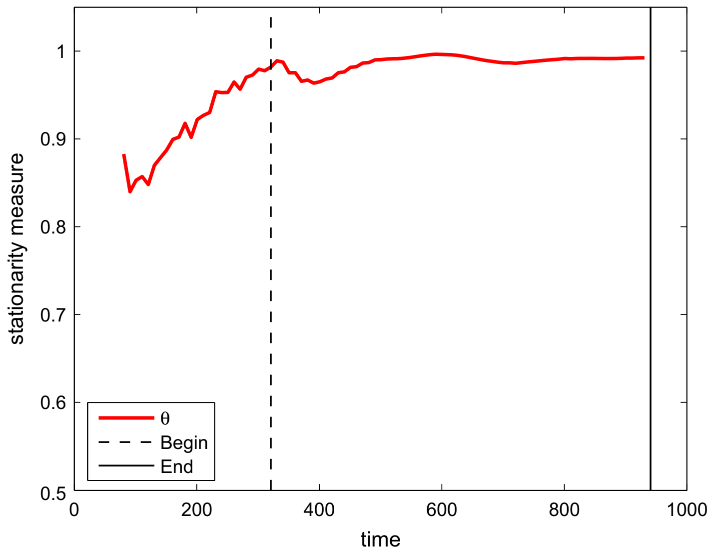

Fig. 3. Stationarity measure θ vs. time of the beginning oscillations.

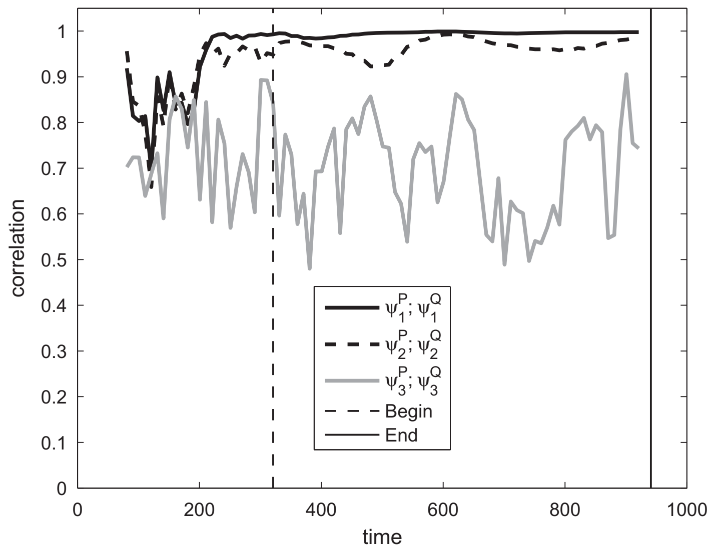

Fig. 4. Correlation of the first three ψ_i^P with ψ_i^Q vs. time of the beginning oscillations.

converge to 1 as the system dynamics evolve from transient motion to stationary movements. Let us note as well that the characteristic functions of larger energy content $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ appear to be more stable, than those characteristic functions of less energy content. Applying Eqs. (15)–(17), stationarity measure θ gives $a_{\text{stat}} = 310$ for the beginning of stationary oscillations. We can verify this result by regarding the set of Figs. 4 and 5. Thus, we find that until around $k = 300$ dynamics (energy) of the coupled oscillator transfer to a single standing wave of the form of the first characteristic function $\boldsymbol{\psi}_1$. During the same period, the first characteristic function $\boldsymbol{\psi}_1$stabilizes, as well.

6.1.2. Disturbed oscillations

In order to analyze the decay of stationary motion, in a second setting, we simulate the oscillator's behavior, subject to a varying excitation amplitude: $f(k) = f_0(k)\cos(-0.01\,k)$. The simulation results are shown in Fig. 6. It is easy to observe stationary motion of $\mathbf{y}(k)$ vanish, but it is difficult to give an exact time for the end of stationary behavior.

With the previous settings of $T_{\min} = 80$ and limits $\theta_{\text{lim}1} = 0.98$ and $\theta_{\text{lim}2} = 0.96$, we again calculate one KLT-basis for each time interval and analyze the dynamics of disturbed oscillations by stationary measure θ and by correlations of the characteristic functions $r\{\boldsymbol{\psi}_i^Q, \boldsymbol{\psi}_i^P\}$, in parallel. The correlation coefficients $r\{\boldsymbol{\psi}_i^Q, \boldsymbol{\psi}_i^P\}$ of the first three characteristic functions

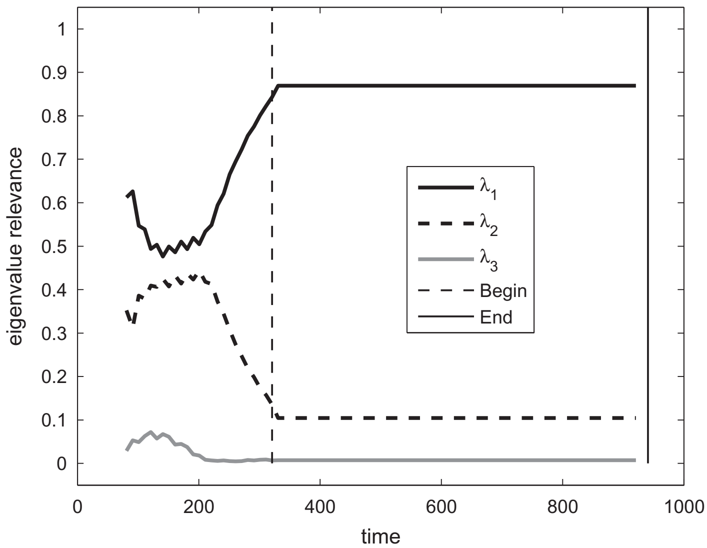

Fig. 5. Normalized first three λ_i vs. time of the beginning oscillations.

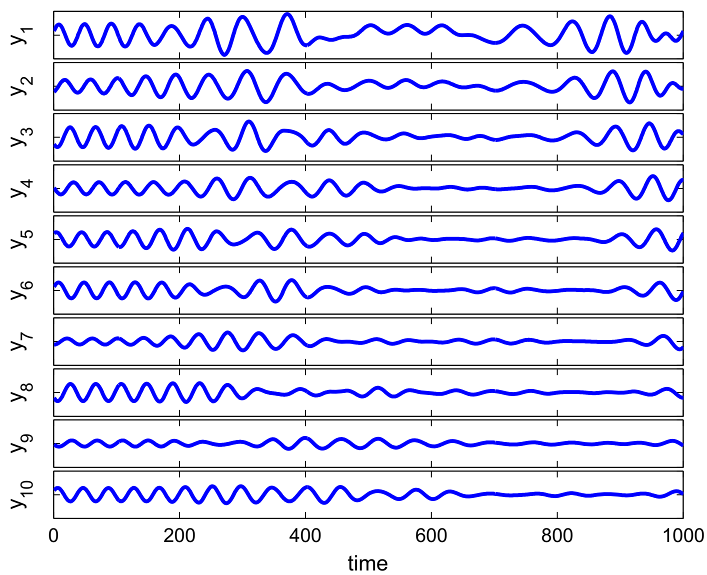

Fig. 6. Simulated dynamics of coupled oscillator: disturbed oscillation.

and the corresponding normalized of the first three eigenvalues $\lambda_i/\sum\lambda_i$ are shown in Figs. 8 and 9, respectively. The stationarity measure θ is displayed in Fig. 7 for $T_{\min}=80$.

It can be seen in Fig. 7 that the stationarity measure θ has larger values at the beginning $k\in[80,110]$ and at the end $k\in[780,900]$ of the simulation time, and thus detects three short intervals of stationary oscillation. This observation corresponds only partly to the correlation coefficients $r\{\boldsymbol{\psi}_i^P,\boldsymbol{\psi}_i^Q\}$ in Fig. 8. While at the beginning, characteristic functions $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$ and $\boldsymbol{\psi}_3$ share values close to one and thus support the findings of θ, at the end of the simulation characteristic function $\boldsymbol{\psi}_1$ show values even below 0.8.

This finding seems to be contradictory to our hypothesis that covariance matrices $\mathbf{C}_{yy}$ and thus θ can be taken as measures of the stability of KLT-bases Ψ in the presence of changing boundary conditions. Nevertheless, we can explain this observation by regarding the corresponding eigenvalues λ_i, Fig. 9: We can find that at the beginning of the simulation, $\boldsymbol{\psi}_1$ contributes most to the system dynamics, $\boldsymbol{\psi}_2$ has little contribution and $\boldsymbol{\psi}_3$ is irrelevant, while at the end of the simulation the system dynamics are governed by $\boldsymbol{\psi}_2$, solely. The stationary measure θ correctly detects the phases of stationary motion at the end of the simulation period, despite smaller values of $r\{\boldsymbol{\psi}_2^P,\boldsymbol{\psi}_2^Q\}$ and $r\{\boldsymbol{\psi}_3^P,\boldsymbol{\psi}_3^Q\}$, because

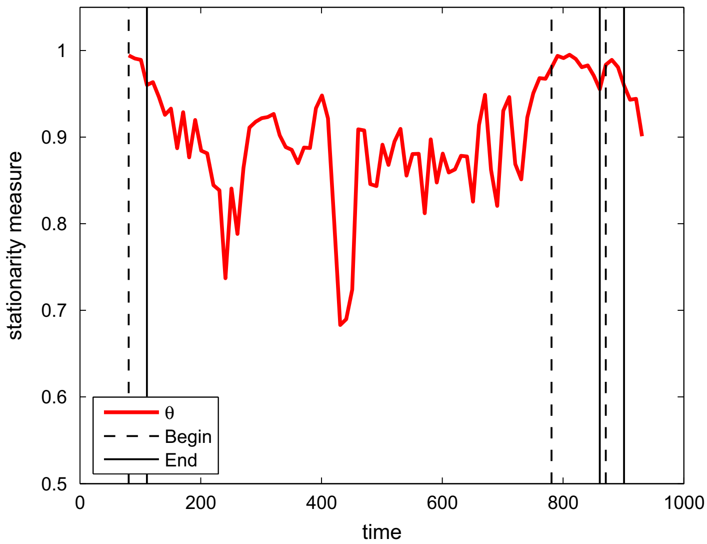

Fig. 7. Stationarity measure θ vs. time of the disturbed oscillations.

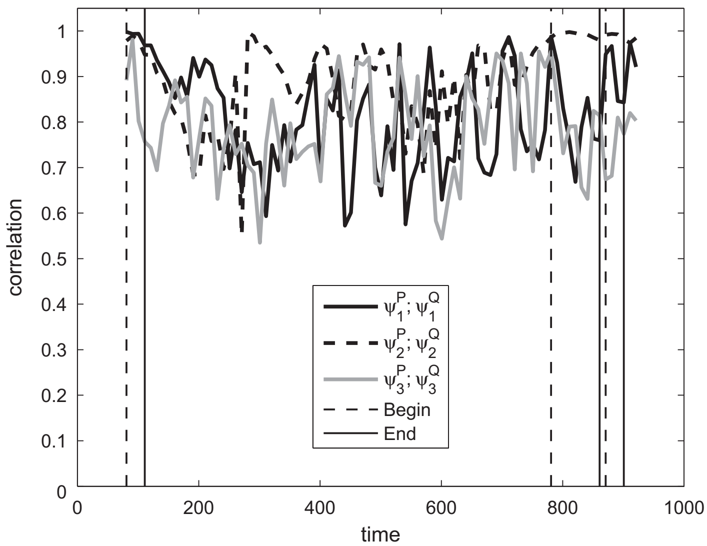

Fig. 8. Correlation of the first three ψ_i^P with ψ_i^Q vs. time of the disturbed oscillations.

characteristic functions $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ change in importance at $k = 280$. This is exactly the time at which θ shows its first significant minimum peak.

6.2. Intermittent behavior of the 'policeman' triple pendulum

In a third analysis, we simulated the highly nonlinear dynamics of the policeman-pendulum that is sketched in Fig. 10. The simulation results in Fig. 11 show that the pendulum undergoes different states of dynamic behavior: While the larger pendulum oscillates rather irregularly β_1, the arms β_2 and β_3 oscillate almost harmonically with varying amplitudes. The most interesting sections of dynamical behavior are around $k \approx 400$, $k \approx 900$, when pendulum 2 shows full rotation and around $k \in [2250, 2700]$, $k \in [2950, 3050]$ and $k \in [3100, 3300]$, when the oscillation amplitude of pendulum 3 exceeds that of pendulum 2.

In order to determine the performance of stationarity measure θ, we compare the results of θ with the correlation coefficients of the most important characteristic functions of the corresponding data sets Y^Q and Y^P. We determine begin

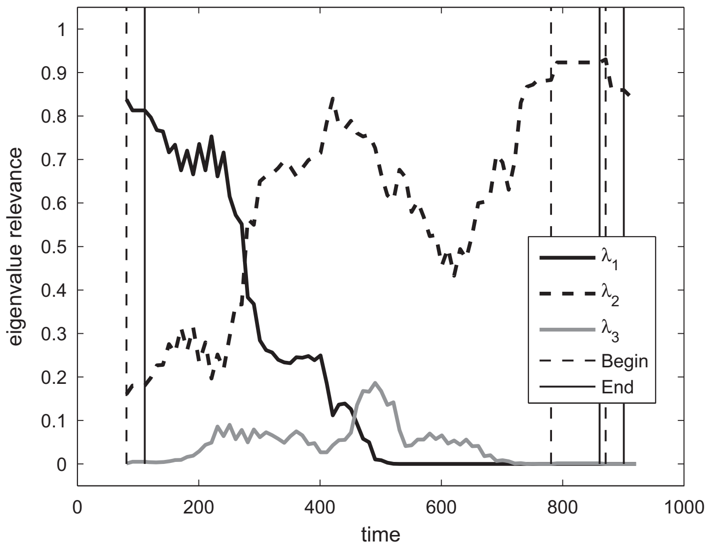

Fig. 9. Normalized first three λ_i vs. time of the disturbed oscillations.

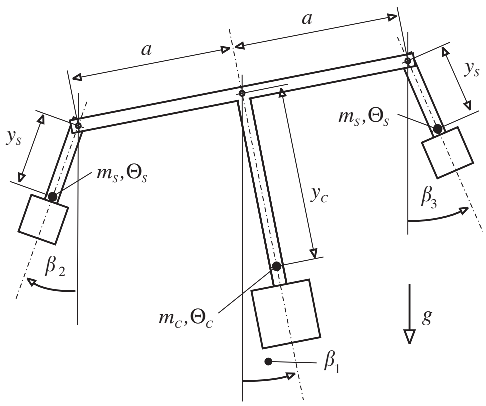

Fig. 10. Policeman-pendulum.

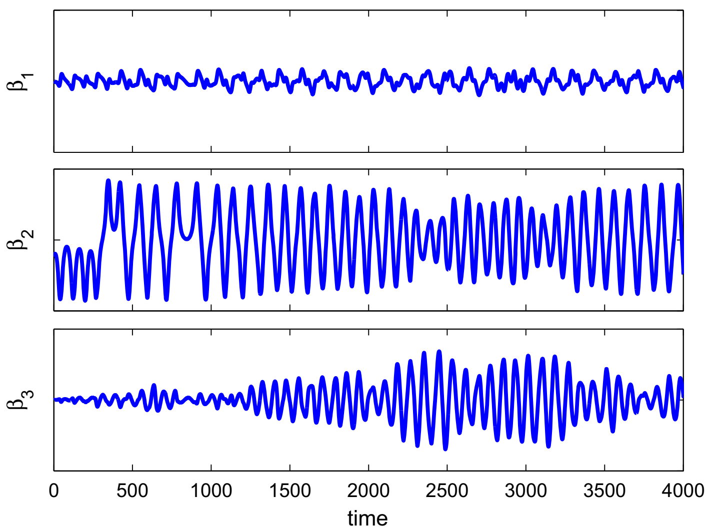

Fig. 11. Simulation results of the pendulum dynamics.

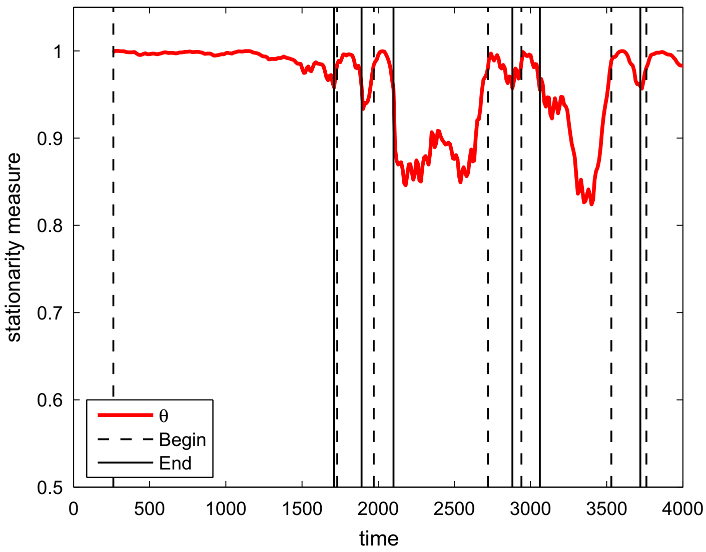

Fig. 12. Stationarity measure θ vs. time of the pendulum dynamics.

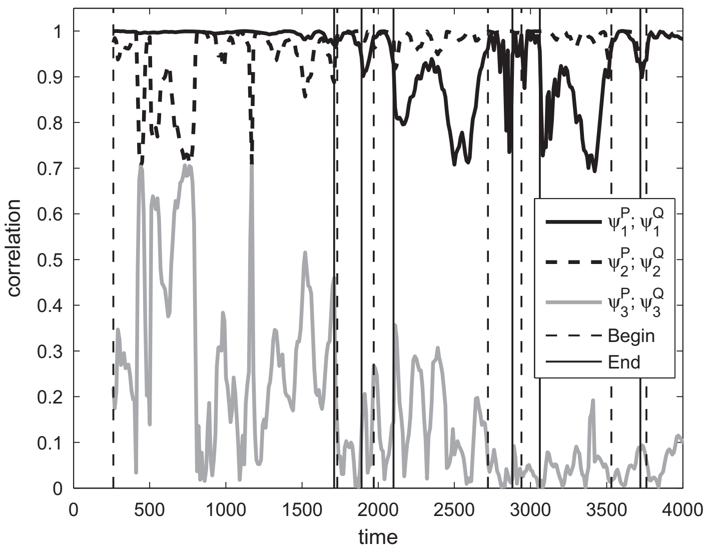

Fig. 13. Correlation of the first three ψ_i^P with ψ_i^Q vs. time of the pendulum dynamics.

a_{stat} and end b_{stat} of stationary motion using Eqs. (15) to (17) and Eqs. (18) to (20), respectively, with limits $\theta_{\lim 1} = 0.98$, $\theta_{\lim 2} = 0.97$ and $T_{\min} = 260$.

The correlation coefficients $r\{\psi_i^Q, \psi_i^P\}$ of the characteristic functions are shown in Fig. 13, the corresponding normalized eigenvalues in Fig. 14 and the stationarity measure θ is displayed in Fig. 12. We can observe in Fig. 12 that the stationarity measure θ is close to 1 for a larger time period $k \in [260, 1700]$ at the beginning of the simulation. Then, θ falls below 0.96 and rises above 0.98 during shorter periods of intermittent dynamics. The changes in the direction of rotation of pendulum 2 at $k \approx 400$, $k \approx 900$ cannot be detected by the stationarity measure θ. However, as far as the performance of a given KLT-basis $\tilde{\Psi}$ is concerned, θ remains a reliable indicator for changes in stationarity, since directions of motion are not relevant, but amplitudes that correspond to the individual energy of the pendulum. During periods $k \in [2250, 2700]$, $k \in [3100, 3300]$ the amplitudes of the oscillations of pendulums 2 and 3 vary continuously. During these intervals, θ correctly shows values of transient dynamics. Only for $k \in [2950, 3050]$ θ reaches values above the limit 0.98 thus detecting stationary motion.

We can check the results of θ by looking at the correlation coefficients $r\{\psi_i^Q, \psi_i^P\}$, Fig. 13, in combination with the corresponding eigenvalues λ_i, Fig. 14: For $k \in [260, 1700]$, the first characteristic function is close to 1, while the correlations of the second and third characteristic functions vary widely. The corresponding eigenvalues λ_i, Fig. 14, reveal that during

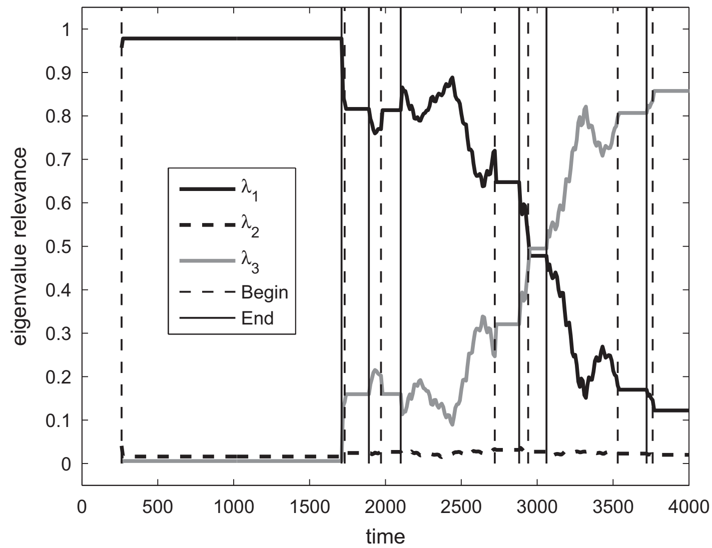

Fig. 14. Eigenvalues λ_i vs. time of the pendulum dynamics.

this period the first characteristic function is paramount. Thus, verifying the results of θ. For $k \in [2250, 2700]$ and $k \in [3100, 3300]$ correlation coefficients $r\{\psi_1^Q, \psi_1^P\}$ and $r\{\psi_3^Q, \psi_3^P\}$ vary extensively. Moreover, the importance of the characteristic functions ψ_1 and ψ_2 change as well (λ_2 becomes larger than λ_1). These results also support the findings of θ. For $k \in [2950, 3050]$ the correlation coefficients $r\{\psi_1^Q, \psi_1^P\}$ and $r\{\psi_2^Q, \psi_2^P\}$ are close to 1. Eigenvalues λ_1, λ_2 show that the corresponding characteristic functions ψ_1 and ψ_2 are equally important. Thus, correlation functions show that reduced KLT-bases $\tilde{\Psi}$ of consecutive intervals I are equivalent and that process $\mathbf{y}(k)$ is stationary.

The analyses of all three examples show that the results of the stationarity measure θ agree with the correlation coefficients of the governing characteristic functions $r\{\psi_i^Q, \psi_i^P\}$ of system dynamics. Thus, stationarity measure θ correctly indicates the stationarity of the major statistical properties of process $\mathbf{y}(k)$, and can thus be used to monitor the performance of a given reduced KLT-basis $\tilde{\Psi}$.

7. Conclusion

The KLT is a favorable tool for model order reduction, since the reduced KLT-basis $\tilde{\Psi}$ is the optimum approximation of $\mathbf{y}(k)$ in the least square sense. However, the KLT is highly sensitivity to changes in the boundary conditions that alter the statistical properties of the dynamics of the system under investigation. Therefore, careful attention has to be paid to the performance of a given reduced KLT-basis $\tilde{\Psi}^Q$ when applied in a model–order–reduction scheme to systems with different settings $\mathbf{y}^P(k)$. In these cases, the given $\tilde{\Psi}^Q$ may not represent the current system dynamics $\mathbf{y}^P(k)$ correctly.

To our knowledge, up to now this effect could neither be calculated a priori nor monitored, efficiently. The performance of the given reduced KLT-basis could only be evaluated by numerically costly secondary calculations of the same problem setting using the original full set of equations. Of course such a procedure would jeopardize the KLT-ansatz itself. Therefore, recent publications either justify the use of one given reduced KLT-basis by arguing with minor changes to the reference setup, or ignore the case sensitivity entirely.

In this paper, we addressed problems of multidimensional oscillating dynamic systems with changing boundary conditions or intermittent dynamic behavior. We introduced a new a priori measure to evaluate the performance of the given reduced KLT-basis $\tilde{\Psi}^Q$ in representing the current behavior of $\mathbf{y}^P(k)$. We define the stationarity measure θ that represents changes in any two consecutive covariance matrices $\mathbf{C}_{yy}(k)$, $k \in [a_1, b_1]$ and $\mathbf{C}_{yy}(k)$, $k \in [a_2, b_2]$ with $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$. Moreover, we could show that stationarity measure θ is correlated to the system dynamics and provides a reasonable measure for the stationarity of the system dynamics.

Thus, we introduced a method to detect necessary updates of given reduced bases $\tilde{\Psi}^Q$ that can be embedded in adaptive model reduction schemes. The only information necessary to calculate the proposed stationarity measure θ is the maximum periodic time.

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